

Supplementary Material for *Differentiable Curl-Noise: Boundary-Respecting Procedural Incompressible Flows Without Discontinuities*

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1 MOTIVATION: CONTINUITY OF ψ'

PROPOSITION A: Let \mathbf{x} be the query point and let $\hat{d}_{s_1}(\mathbf{x}), \hat{d}_{s_2}(\mathbf{x}), \dots, \hat{d}_{s_n}(\mathbf{x})$ be the sorted LSE distance function defined in Section 4.2. Assume the user-defined potential $\psi(\mathbf{x})$ and $cp(\mathbf{x})$ are at least C^1 differentiable. If $d_0(\mathbf{x}) = \min\{\hat{d}_{s_2}(\mathbf{x}), d_0\}$, then

$$\text{Multiplicative Ramping: } \psi'(\mathbf{x}) = \alpha(\mathbf{x})\psi(\mathbf{x}) + (1 - \alpha(\mathbf{x}))\psi_g(\mathbf{x}), \quad (1)$$

and

$$\text{Additive Ramping: } \psi'(\mathbf{x}) = \psi(\mathbf{x}) + (\psi_g(\mathbf{x}) - \psi(cp(\mathbf{x}))(1 - \alpha(\mathbf{x})) \quad (2)$$

are both continuous, where

$$\alpha(\mathbf{x}) = \text{ramp}\left(\frac{\bar{d}^{-1}(\mathbf{x})}{d_0(\mathbf{x})}\right), \quad \text{and} \quad \text{ramp}(r) = \begin{cases} 1 & \text{if } r \geq 1 \\ \frac{15}{8}r - \frac{10}{8}r^3 + \frac{3}{8}r^5 & \text{if } -1 < r < 1 \\ -1 & \text{if } r \leq -1 \end{cases}$$

PROOF. We first make the following five observations:

Observation 1: Directly following their definitions, we know $\text{ramp}(r) \in C^1$ and

$$d(\mathbf{x}) \leq \bar{d}^{-1}(\mathbf{x}) = \frac{\sum_{i=1}^n \hat{d}_i(\mathbf{x}) \exp(-b\hat{d}_i(\mathbf{x}))}{\sum_{i=1}^n \exp(-b\hat{d}_i(\mathbf{x}))} = \frac{\sum_{i=1}^n \hat{d}_{s_i}(\mathbf{x}) \exp(-b\hat{d}_{s_i}(\mathbf{x}))}{\sum_{i=1}^n \exp(-b\hat{d}_{s_i}(\mathbf{x}))} \in C^1.$$

Observation 2: For all \mathbf{x} located on the equidistant curve of the closest two obstacles (i.e., when $d(\mathbf{x}) = \hat{d}_{s_1}(\mathbf{x}) = \hat{d}_{s_2}(\mathbf{x})$), we have $1 - \alpha(\mathbf{x}) = 0$.

Proof of observation 2: Since $\hat{d}_{s_1}(\mathbf{x}) \leq \hat{d}_{s_2}(\mathbf{x})$, there are only three possible cases regarding the order of $\hat{d}_{s_1}(\mathbf{x}), \hat{d}_{s_2}(\mathbf{x})$ and d_0 . We analyze them one by one.

Case 1: If $d_0 \leq \hat{d}_{s_1}(\mathbf{x}) \leq \hat{d}_{s_2}(\mathbf{x})$, then $d_0(\mathbf{x}) = \min\{\hat{d}_{s_2}(\mathbf{x}), d_0\} = d_0$ and $\frac{\bar{d}^{-1}(\mathbf{x})}{d_0(\mathbf{x})} \geq \frac{d(\mathbf{x})}{d_0} \geq 1$.

Case 2: If $\hat{d}_{s_1}(\mathbf{x}) \leq d_0 \leq \hat{d}_{s_2}(\mathbf{x})$, then $d_0(\mathbf{x}) = \min\{\hat{d}_{s_2}(\mathbf{x}), d_0\} = d_0$. When $\hat{d}_{s_1}(\mathbf{x}) = \hat{d}_{s_2}(\mathbf{x})$, we must have $\hat{d}_{s_1}(\mathbf{x}) = d_0 = \hat{d}_{s_2}(\mathbf{x})$. So, $\frac{\bar{d}^{-1}(\mathbf{x})}{d_0(\mathbf{x})} \geq \frac{d(\mathbf{x})}{d_0} = \frac{d(\mathbf{x})}{\hat{d}_{s_2}(\mathbf{x})} \geq 1$.

Case 3: If $\hat{d}_{s_1}(\mathbf{x}) \leq \hat{d}_{s_2}(\mathbf{x}) \leq d_0$, then $d_0(\mathbf{x}) = \min\{\hat{d}_{s_2}(\mathbf{x}), d_0\} = \hat{d}_{s_2}(\mathbf{x})$. When $\hat{d}_{s_1}(\mathbf{x}) = \hat{d}_{s_2}(\mathbf{x})$, we

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have $\frac{\bar{d}^1(\mathbf{x})}{d_0(\mathbf{x})} \geq \frac{d(\mathbf{x})}{\hat{d}_{s_2}(\mathbf{x})} = 1$.

Notice that, in all three cases, we showed that $\frac{\bar{d}^1(\mathbf{x})}{d_0(\mathbf{x})} \geq 1$ when $\hat{d}_{s_1}(\mathbf{x}) = \hat{d}_{s_2}(\mathbf{x})$. So,

$$\alpha(\mathbf{x}) = \text{ramp}\left(\frac{\bar{d}^1(\mathbf{x})}{d_0(\mathbf{x})}\right) = 1 \implies 1 - \alpha(\mathbf{x}) = 0.$$

□

Observation 3: Eq. 1 and 2 are both of the general form

$$\psi'(\mathbf{x}) = \psi(\mathbf{x}) + (\psi_g(\mathbf{x}) - \psi(\mathbf{m}(\mathbf{x}))(1 - \alpha(\mathbf{x})). \quad (3)$$

We can rewrite Eq. 1 as

$$\psi'(\mathbf{x}) = \alpha(\mathbf{x})\psi(\mathbf{x}) + (1 - \alpha(\mathbf{x}))\psi_g(\mathbf{x}) = \psi(\mathbf{x}) + (\psi_g(\mathbf{x}) - \psi(\mathbf{x}))(1 - \alpha(\mathbf{x})),$$

where $\mathbf{m}(\mathbf{x}) = \mathbf{x}$. It is clear that $\mathbf{m}(\mathbf{x}) = cp(\mathbf{x})$ for Eq. 2.

Observation 4: $\hat{d}_{s_2}(\mathbf{x})$ is a strictly positive C^0 function. It is not differentiable over \mathcal{D} wherever the second closest object to \mathbf{x} changes. It is strictly positive since \mathbf{x} is in the exterior of all objects. Therefore, $d_0(\mathbf{x})$ is a strictly positive C^0 function.

Observation 5: The user defined potential evaluated at the geometric center of the closest object to \mathbf{x} , i.e., $\psi_g(\mathbf{x})$, is a Heaviside step function. It is discontinuous wherever the closest object to \mathbf{x} changes. That is, all discontinuities of $\psi_g(\mathbf{x})$ locate on the equidistant curve between the closest and second closest object to \mathbf{x} , where $\hat{d}_{s_1}(\mathbf{x}) = \hat{d}_{s_2}(\mathbf{x})$. Moreover, as all the objects are defined in a finite and bounded domain \mathcal{D} , then ψ_g is bounded. Therefore, $\exists M \geq 0$ such that $|\psi_g| \leq M$.

Under the assumption that $\psi(\mathbf{x}), \mathbf{m}(\mathbf{x}) \in C^1$, and relying on the preceding Observations, it suffices to show that Eq. 3 is continuous at all \mathbf{x} that satisfy $\hat{d}_{s_1}(\mathbf{x}) = \hat{d}_{s_2}(\mathbf{x})$.

Take arbitrary $\epsilon > 0$. Set $\epsilon_0 = \frac{1}{3}\epsilon$. Let \mathbf{x}_0 be any query point such that $\hat{d}_{s_1}(\mathbf{x}_0) = \hat{d}_{s_2}(\mathbf{x}_0)$.

Since $\psi(\mathbf{x}) \in C^1$, then $\forall \epsilon_0 > 0, \exists \delta_1 > 0$ such that

$$|\mathbf{x} - \mathbf{x}_0| < \delta_1 \implies |\psi(\mathbf{x}) - \psi(\mathbf{x}_0)| < \epsilon_0. \quad (4)$$

By *Observation 1* and *Observation 4*, we know that $\frac{\bar{d}^1(\mathbf{x})}{d_0(\mathbf{x})}$ is continuous, and so is $\text{ramp}\left(\frac{\bar{d}^1(\mathbf{x})}{d_0(\mathbf{x})}\right)$. By construction, $\alpha(\mathbf{x}) = \text{ramp}\left(\frac{\bar{d}^1(\mathbf{x})}{d_0(\mathbf{x})}\right)$, and $1 - \alpha(\mathbf{x})$ are both continuous. Then $\forall \frac{\epsilon_0}{M} > 0, \exists \delta_2 > 0$ such that

$$|\mathbf{x} - \mathbf{x}_0| < \delta_2 \implies \left| (1 - \alpha(\mathbf{x})) - \underbrace{(1 - \alpha(\mathbf{x}_0))}_0 \right| = |1 - \alpha(\mathbf{x})| < \frac{\epsilon_0}{M}. \quad (5)$$

by *Observation 2*

By assumption, as $\psi(\mathbf{x}), \mathbf{m}(\mathbf{x}) \in C^1$, then $\psi(\mathbf{m}(\mathbf{x}))(1 - \alpha(\mathbf{x})) \in C^1$. Then $\forall \epsilon_0 > 0, \exists \delta_2 > 0$ such that

$$|\mathbf{x} - \mathbf{x}_0| < \delta_3 \implies \left| \psi(\mathbf{m}(\mathbf{x}))(1 - \alpha(\mathbf{x})) - \psi(\mathbf{m}(\mathbf{x}_0)) \underbrace{(1 - \alpha(\mathbf{x}_0))}_0 \right|$$

by *Observation 2*

$$= |\psi(\mathbf{m}(\mathbf{x}))(1 - \alpha(\mathbf{x}))| < \epsilon_0. \quad (6)$$

Take $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, and assume $\|\mathbf{x} - \mathbf{x}_0\| < \delta$, then

$$\begin{aligned}
& |\psi'(\mathbf{x}) - \psi'(\mathbf{x}_0)| \\
&= |\psi(\mathbf{x}) + (\psi_g(\mathbf{x}) - \psi(\mathbf{m}(\mathbf{x}))(1 - \alpha(\mathbf{x})) - \psi(\mathbf{x}_0) - (\psi_g(\mathbf{x}_0) - \psi(\mathbf{m}(\mathbf{x}_0)) \underbrace{(1 - \alpha(\mathbf{x}_0))}_0)| \\
&\hspace{15em} \text{by Observation 2} \\
&= |\psi(\mathbf{x}) + (\psi_g(\mathbf{x}) - \psi(\mathbf{m}(\mathbf{x}))(1 - \alpha(\mathbf{x})) - \psi(\mathbf{x}_0)| \\
&\leq |\psi(\mathbf{x}) - \psi(\mathbf{x}_0)| + |\psi_g(\mathbf{x})||1 - \alpha(\mathbf{x})| + |\psi(\mathbf{m}(\mathbf{x}))(1 - \alpha(\mathbf{x}))| \\
&\leq \epsilon_0 + M \frac{\epsilon_0}{M} + \epsilon_0 \quad \text{by Eq. 4, 5, 6, and Observation 5} \\
&= 3\epsilon_0 = \epsilon.
\end{aligned}$$

Therefore, we showed that any $\psi'(\mathbf{x})$ having the form of Eq. 3 is continuous. Hence, by *Observation 3*, Eq. 1 and 2 are both continuous given that $\psi(\mathbf{x}), \mathbf{m}(\mathbf{x}) \in C^1$. \square

2 ALGORITHM 1: DIFFERENTIABILITY OF ψ'

PROPOSITION B: Let \mathbf{x} be the query point and let $\hat{d}_{s_1}(\mathbf{x}), \hat{d}_{s_2}(\mathbf{x}), \dots, \hat{d}_{s_n}(\mathbf{x})$ be the sorted LSE distance functions defined in Section 4.2. Assume the user-defined potential $\psi(\mathbf{x})$ and $cp(\mathbf{x})$ are at least C^1 differentiable. If $d_0(\mathbf{x})$ is approximated using the C^1 construction in Algorithm 1, then

$$\text{Eq. 1: } \psi'(\mathbf{x}) = \alpha(\mathbf{x})\psi(\mathbf{x}) + (1 - \alpha(\mathbf{x}))\psi_g(\mathbf{x}), \quad \text{and}$$

$$\text{Eq. 2: } \psi'(\mathbf{x}) = \psi(\mathbf{x}) + (\psi_g(\mathbf{x}) - \psi(cp(\mathbf{x}))(1 - \alpha(\mathbf{x})))$$

are both continuously differentiable, where

$$\alpha(\mathbf{x}) = \text{ramp}\left(\frac{\bar{d}^{-1}(\mathbf{x})}{d_0(\mathbf{x})}\right), \quad \text{and} \quad \text{ramp}(r) = \begin{cases} 1 & \text{if } r \geq 1 \\ \frac{15}{8}r - \frac{10}{8}r^3 + \frac{3}{8}r^5 & \text{if } -1 < r < 1 \\ -1 & \text{if } r \leq -1 \end{cases}$$

PROOF. Notice that *Observations 1, 3, 5* in PROPOSITION A directly hold true under this setup, and *Observation 4* is also true when a is large enough. Following PROPOSITION A and based on the fact that $d_0(\mathbf{x})$ is now C^1 , we know that $\psi'(\mathbf{x})$ can be continuous but not differentiable only when the closest object to \mathbf{x} changes. For all other query points, $\psi'(\mathbf{x})$ is C^1 since $\psi_g(\mathbf{x})$ does not touch the discontinuity and thus all components are C^1 . So, we only need to prove

$$\psi'(\mathbf{x}) = \psi(\mathbf{x}) + (\psi_g(\mathbf{x}) - \psi(\mathbf{m}(\mathbf{x}))(1 - \alpha(\mathbf{x})),$$

is continuously differentiable, where we further assume $\mathbf{m}(\mathbf{x})$ is at least C^1 . Therefore, it suffices to show that both $\frac{\partial \psi'(\mathbf{x})}{\partial x}$ and $\frac{\partial \psi'(\mathbf{x})}{\partial y}$ are continuous for all \mathbf{x} such that $\hat{d}_{s_1}(\mathbf{x}) = \hat{d}_{s_2}(\mathbf{x})$. In this proof, we will show that $\frac{\partial \psi'(\mathbf{x})}{\partial x}$ is continuous; the proof for differentiating y is similar.

Now, we want to show *Observation 2* in PROPOSITION A is still true when $d_0(x)$ is calculated using Algorithm 1.

Observation 2: For all \mathbf{x} located on the equidistant curve of the closest two obstacles, (i.e., when $d(\mathbf{x}) = \hat{d}_{s_1}(\mathbf{x}) = \hat{d}_{s_2}(\mathbf{x})$), we have $1 - \alpha(\mathbf{x}) = 0$ and $\frac{\partial \alpha}{\partial x}(\mathbf{x}) = \frac{\partial \alpha}{\partial y}(\mathbf{x}) = 0$.

Proof of observation 2: From Section 4.2.1, we know that our differentiable $d_0(\mathbf{x})$ underestimates $\hat{d}_{s_2}(\mathbf{x})$ and overestimates $d(\mathbf{x}) = \hat{d}_{s_1}(\mathbf{x})$. Since $\hat{d}_{s_1}(\mathbf{x}) \leq \hat{d}_{s_2}(\mathbf{x})$, there are still three possible cases regarding the ordering of $\hat{d}_{s_1}(\mathbf{x}), \hat{d}_{s_2}(\mathbf{x})$ and d_0 . We analyze them one by one.

Case 1: $d_0 \leq \hat{d}_{s_1}(\mathbf{x}) \leq \hat{d}_{s_2}(\mathbf{x})$. When $\hat{d}_{s_1}(\mathbf{x}) = \hat{d}_{s_2}(\mathbf{x})$, we have $d_0 \leq \hat{d}_{s_1}(\mathbf{x}) = \hat{d}_{s_2}(\mathbf{x})$. No matter which one of $\bar{d}^2(\mathbf{x})$ or d_0 is smaller, we will always have $\bar{d}^{-1}(\mathbf{x}) \geq d_0(\mathbf{x})$ since $\bar{d}^{-1}(\mathbf{x}) \geq d(\mathbf{x}) = \hat{d}_{s_1}(\mathbf{x}) = \hat{d}_{s_2}(\mathbf{x}) \geq \bar{d}^2(\mathbf{x}) \geq d_0(\mathbf{x})$, and $d_0 \geq d_0(\mathbf{x})$ (LSE approximation of $\min\{\cdot\}$ is an underestimator).

Therefore $\frac{\bar{d}^1(\mathbf{x})}{d_0(\mathbf{x})} \geq 1$.

Case 2: $\hat{d}_{s_1}(\mathbf{x}) \leq d_0 \leq \hat{d}_{s_2}(\mathbf{x})$. When $\hat{d}_{s_1}(\mathbf{x}) = \hat{d}_{s_2}(\mathbf{x})$, we must have $\hat{d}_{s_1}(\mathbf{x}) = d_0 = \hat{d}_{s_2}(\mathbf{x})$. It follows that $\bar{d}^1(\mathbf{x}) \geq d(\mathbf{x}) = \hat{d}_{s_1}(\mathbf{x}) = d_0 = \hat{d}_{s_2}(\mathbf{x}) \geq \tilde{d}^2(\mathbf{x}) \geq d_0(\mathbf{x})$, which implies $\frac{\bar{d}^1(\mathbf{x})}{d_0(\mathbf{x})} \geq 1$.

Case 3: $\hat{d}_{s_1}(\mathbf{x}) \leq \hat{d}_{s_2}(\mathbf{x}) \leq d_0$. When $\hat{d}_{s_1}(\mathbf{x}) = \hat{d}_{s_2}(\mathbf{x})$, we have $\hat{d}_{s_1}(\mathbf{x}) = \hat{d}_{s_2}(\mathbf{x}) \leq d_0$. Then $\bar{d}^1(\mathbf{x}) \geq d(\mathbf{x}) = \hat{d}_{s_1}(\mathbf{x}) = \hat{d}_{s_2}(\mathbf{x}) \geq \tilde{d}^2(\mathbf{x}) \geq d_0(\mathbf{x})$. Hence, $\frac{\bar{d}^1(\mathbf{x})}{d_0(\mathbf{x})} \geq 1$.

Notice that in all three cases, we showed that $\frac{\bar{d}^1(\mathbf{x})}{d_0(\mathbf{x})} \geq 1$, given that $\hat{d}_{s_1}(\mathbf{x}) = \hat{d}_{s_2}(\mathbf{x})$. So,

$$\alpha(\mathbf{x}) = \text{ramp}\left(\frac{\bar{d}^1(\mathbf{x})}{d_0(\mathbf{x})}\right) = 1 \implies 1 - \alpha(\mathbf{x}) = 0, \quad \text{and} \quad (7)$$

$$\nabla\alpha(\mathbf{x}) = \underbrace{\text{ramp}'\left(\frac{\bar{d}^1(\mathbf{x})}{d_0(\mathbf{x})}\right)}_{=0} \cdot \nabla\left(\frac{\bar{d}^1(\mathbf{x})}{d_0(\mathbf{x})}\right) = 0 \implies \frac{\partial\alpha}{\partial x} = \frac{\partial\alpha}{\partial y} = 0 \quad (8)$$

□

We make one last observation based on the discussion in Section 4.2.1.

Observation 6: $\forall \mathbf{x} \in \mathcal{D}, \nabla\psi_g(\mathbf{x})(1 - \alpha(\mathbf{x})) = 0$.

Proof of Observation 6: In Section 4.2.1, we mentioned that the first order partial derivatives of $\psi_g(\mathbf{x})$ are linear combinations of Dirac delta functions, where the nonzero values occur for \mathbf{x} on the equidistant curves between the two closest objects to \mathbf{x} . So, if \mathbf{x} is not on any equidistant curve causing a change of the closest object, then $\nabla\psi_g(\mathbf{x}) = 0$. Otherwise, as we proved in *Observation 2* that $(1 - \alpha(\mathbf{x})) = 0$. Therefore, we will always have $\nabla\psi_g(\mathbf{x})(1 - \alpha(\mathbf{x})) = 0$. □

Let \mathbf{x}_0 be any query point such that $\hat{d}_{s_1}(\mathbf{x}_0) = \hat{d}_{s_2}(\mathbf{x}_0)$.

Here, based on the general form (Eq. 3), we write down the partial derivative of $\psi'(\mathbf{x})$ with respect to x .

$$\frac{\partial\psi'}{\partial x}(\mathbf{x}) = \frac{\partial\psi}{\partial x}(\mathbf{x}) + \left(\frac{\partial\psi_g}{\partial x}(\mathbf{x}) + \frac{\partial\psi(\mathbf{m}(\mathbf{x}))}{\partial x}\right)(1 - \alpha(\mathbf{x})) - (\psi_g(\mathbf{x}) + \psi(\mathbf{m}(\mathbf{x})))\frac{\partial\alpha}{\partial x}(\mathbf{x}) \quad (9)$$

When $\mathbf{x} = \mathbf{x}_0$, then Eq. 9 is

$$\begin{aligned} \frac{\partial\psi'}{\partial x}(\mathbf{x}_0) &= \frac{\partial\psi}{\partial x}(\mathbf{x}_0) + \left(\frac{\partial\psi_g}{\partial x}(\mathbf{x}_0) + \frac{\partial\psi(\mathbf{m}(\mathbf{x}_0))}{\partial x}\right)(1 - \alpha(\mathbf{x}_0)) - (\psi_g(\mathbf{x}_0) + \psi(\mathbf{m}(\mathbf{x}_0)))\frac{\partial\alpha}{\partial x}(\mathbf{x}_0) \\ &= \frac{\partial\psi}{\partial x}(\mathbf{x}_0) \quad \text{By Observation 2, } 1 - \alpha(\mathbf{x}_0) = 0 \text{ and } \frac{\partial\alpha}{\partial x}(\mathbf{x}_0) = 0. \end{aligned} \quad (10)$$

Take arbitrary $\epsilon > 0$. Set $\epsilon_0 = \frac{1}{4}\epsilon$.

Since $\psi(\mathbf{x}) \in C^1$, then $\forall \epsilon_0 > 0, \exists \delta_1 > 0$ such that

$$|\mathbf{x} - \mathbf{x}_0| < \delta_1 \implies \left|\frac{\partial\psi}{\partial x}(\mathbf{x}) - \frac{\partial\psi}{\partial x}(\mathbf{x}_0)\right| < \epsilon_0. \quad (11)$$

Since $\alpha(\mathbf{x}) \in C^1$, then $\forall \frac{\epsilon_0}{M} > 0, \exists \delta_2 > 0$, where M is the upper bound of $\psi_g(\mathbf{x})$ in *Observation 5*, such that

$$|\mathbf{x} - \mathbf{x}_0| < \delta_2 \implies \left|\frac{\partial\alpha}{\partial x}(\mathbf{x}) - \underbrace{\frac{\partial\alpha}{\partial x}(\mathbf{x}_0)}_0\right| = \left|\frac{\partial\alpha}{\partial x}(\mathbf{x})\right| < \frac{\epsilon_0}{M}. \quad (12)$$

by *Observation 2*

By assumption, as $\psi(\mathbf{x}), \mathbf{m}(\mathbf{x}) \in C^1$, then $\frac{\partial \psi(\mathbf{m}(\mathbf{x}))}{\partial \mathbf{x}}(1 - \alpha(\mathbf{x}))$ is continuous. Then $\forall \epsilon_0 > 0, \exists \delta_3 > 0$ such that

$$\begin{aligned} |\mathbf{x} - \mathbf{x}_0| < \delta_3 &\implies \left| \frac{\partial \psi(\mathbf{m}(\mathbf{x}))}{\partial \mathbf{x}}(1 - \alpha(\mathbf{x})) - \frac{\partial \psi(\mathbf{m}(\mathbf{x}_0))}{\partial \mathbf{x}} \underbrace{(1 - \alpha(\mathbf{x}_0))}_{\text{by Observation 2}} \right| \\ &= \left| \frac{\partial \psi(\mathbf{m}(\mathbf{x}))}{\partial \mathbf{x}}(1 - \alpha(\mathbf{x})) \right| < \epsilon_0. \end{aligned} \quad (13)$$

For a similar reason, $\psi(\mathbf{m}(\mathbf{x})) \frac{\partial \alpha}{\partial \mathbf{x}}(\mathbf{x})$ is also continuous. So, $\forall \epsilon_0 > 0, \exists \delta_4 > 0$ such that

$$\begin{aligned} |\mathbf{x} - \mathbf{x}_0| < \delta_3 &\implies \left| \psi(\mathbf{m}(\mathbf{x})) \frac{\partial \alpha}{\partial \mathbf{x}}(\mathbf{x}) - \psi(\mathbf{m}(\mathbf{x}_0)) \underbrace{\frac{\partial \alpha}{\partial \mathbf{x}}(\mathbf{x}_0)}_{\text{by Observation 2}} \right| \\ &= \left| \psi(\mathbf{m}(\mathbf{x})) \frac{\partial \alpha}{\partial \mathbf{x}}(\mathbf{x}) \right| < \epsilon_0. \end{aligned} \quad (14)$$

Take $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$, and assume $|\mathbf{x} - \mathbf{x}_0| < \delta$, then

$$\begin{aligned} &\left| \frac{\partial \psi'}{\partial \mathbf{x}}(\mathbf{x}) - \frac{\partial \psi'}{\partial \mathbf{x}}(\mathbf{x}_0) \right| \\ &= \left| \frac{\partial \psi}{\partial \mathbf{x}}(\mathbf{x}) + \left(\frac{\partial \psi_g(\mathbf{x})}{\partial \mathbf{x}} + \frac{\partial \psi(\mathbf{m}(\mathbf{x}))}{\partial \mathbf{x}} \right) (1 - \alpha(\mathbf{x})) - (\psi_g(\mathbf{x}) + \psi(\mathbf{m}(\mathbf{x}))) \frac{\partial \alpha}{\partial \mathbf{x}}(\mathbf{x}) - \frac{\partial \psi}{\partial \mathbf{x}}(\mathbf{x}_0) \right| \\ &\quad \text{Sub in Eq. 9 and Eq. 10} \\ &\leq \left| \frac{\partial \psi}{\partial \mathbf{x}}(\mathbf{x}) - \frac{\partial \psi}{\partial \mathbf{x}}(\mathbf{x}_0) \right| + \underbrace{\left| \frac{\partial \psi_g(\mathbf{x})}{\partial \mathbf{x}} (1 - \alpha(\mathbf{x})) \right|}_{\text{by Observation 6}} + \left| \frac{\partial \psi(\mathbf{m}(\mathbf{x}))}{\partial \mathbf{x}} (1 - \alpha(\mathbf{x})) \right| + \\ &\quad \left| (\psi_g(\mathbf{x}) + \psi(\mathbf{m}(\mathbf{x}))) \frac{\partial \alpha}{\partial \mathbf{x}}(\mathbf{x}) \right| \\ &\leq \left| \frac{\partial \psi}{\partial \mathbf{x}}(\mathbf{x}) - \frac{\partial \psi}{\partial \mathbf{x}}(\mathbf{x}_0) \right| + \left| \frac{\partial \psi(\mathbf{m}(\mathbf{x}))}{\partial \mathbf{x}} (1 - \alpha(\mathbf{x})) \right| + |\psi_g(\mathbf{x})| \left| \frac{\partial \alpha}{\partial \mathbf{x}}(\mathbf{x}) \right| + \left| \psi(\mathbf{m}(\mathbf{x})) \frac{\partial \alpha}{\partial \mathbf{x}}(\mathbf{x}) \right| \\ &\leq \epsilon_0 + \epsilon_0 + M \frac{\epsilon_0}{M} + \epsilon_0 \quad \text{By Eq. 11, 12, 13, 14 and Observation 5} \\ &= 4\epsilon_0 = \epsilon. \end{aligned}$$

Hence, we showed that $\frac{\partial \psi'}{\partial \mathbf{x}}$ is continuous even when the closest object to \mathbf{x} changes. A similar conclusion can be drawn for $\frac{\partial \psi'}{\partial \mathbf{y}}$. Therefore, we know that $\psi'(\mathbf{x}) \in C^1$ for all \mathbf{x} such that $\hat{d}_{s_1}(\mathbf{x}) = \hat{d}_{s_2}(\mathbf{x})$. As for all other query points, $\psi'(\mathbf{x})$ is already C^1 , we conclude that $\psi'(\mathbf{x}) \in C^1, \forall \mathbf{x} \in \mathcal{D}$.

3 EXTEND TO 3D: A RAY MARCHING ALGORITHM

CLAIM: Take arbitrary $i \in [n]$. Consider the i^{th} object, O_i , defined in the domain of interest \mathcal{D} . Let F_i be the set of primitives composing O_i . If $\frac{\partial^2 w_\ell}{\partial \phi_\ell^2} = 0, \forall \ell \in F_i$, then $\nabla d(\mathbf{x}) \in C^1$.

PROOF. We need $\nabla d(\mathbf{x})$ to be differentiable if we want the updating step of the Ray Marching algorithm to be differentiable. However, the design of weight function by Madan and Levin [2022] only guarantees $\nabla \hat{d}_i(\mathbf{x}) \in C^0$, and thus $\nabla d(\mathbf{x}) \in C^0$ (by the construction of our Equation 7 in the paper). Hence, the key step is to design a continuously differentiable $\nabla \hat{d}_i(\mathbf{x})$. To make sure

$\nabla \hat{d}_i(\mathbf{x}) \in C^1$, we need to confine the second order behaviour of $w_\ell(\mathbf{x})$. We first reiterate the expression for $\nabla \hat{d}_i(\mathbf{x})$ in our notation (Equation 4 in Madan and Levin [2022]):

$$\nabla \hat{d}_i(\mathbf{x}) = \frac{\sum_{f_\ell \in F_i} w_\ell(\mathbf{x}) \exp(-ad_\ell^*) \nabla d_\ell^* - \frac{1}{a} \exp(-ad_\ell^*) \nabla w_\ell(\mathbf{x})}{\sum_{f_\ell \in F_i} w_\ell(\mathbf{x}) \exp(-ad_\ell^*)}.$$

Following the nature of the SC Exterior Mapping, we are only querying the points in the domain \mathcal{D} that is also in the exterior of the objects. Then, $\nabla d_\ell^* \in C^1$ holds for all such query points. Therefore, taking a closer look at the expression for $\nabla \hat{d}_i(\mathbf{x})$, we realized that each component of $\nabla \hat{d}_i(\mathbf{x})$ is already C^1 , except for $\nabla w_\ell(\mathbf{x})$. Let us write down $\nabla^2 w_\ell(\mathbf{x})$ explicitly:

$$\begin{aligned} \nabla^2 w_\ell(\mathbf{x}) &= \frac{\partial}{\partial \mathbf{x}} \left(\frac{\partial \pi_\ell}{\partial \mathbf{x}} \frac{\partial \phi_\ell}{\partial \pi_\ell} \frac{\partial w_\ell}{\partial \phi_\ell} \right) \\ &= \frac{\partial}{\partial \mathbf{x}} \left(\frac{\partial \pi_\ell}{\partial \mathbf{x}} \right) \frac{\partial \phi_\ell}{\partial \pi_\ell} \frac{\partial w_\ell}{\partial \phi_\ell} + \frac{\partial}{\partial \mathbf{x}} \left(\frac{\partial \phi_\ell}{\partial \pi_\ell} \right) \frac{\partial \pi_\ell}{\partial \mathbf{x}} \frac{\partial w_\ell}{\partial \phi_\ell} + \frac{\partial}{\partial \mathbf{x}} \left(\frac{\partial w_\ell}{\partial \phi_\ell} \right) \frac{\partial \pi_\ell}{\partial \mathbf{x}} \frac{\partial \phi_\ell}{\partial \pi_\ell} \\ &= \frac{\partial}{\partial \mathbf{x}} \left(\frac{\partial w_\ell}{\partial \phi_\ell} \right) \frac{\partial \pi_\ell}{\partial \mathbf{x}} \frac{\partial \phi_\ell}{\partial \pi_\ell} \quad (\text{since } \frac{\partial w_\ell}{\partial \phi_\ell} = 0 \text{ by Madan and Levin [2022]}) \\ &= \frac{\partial^2 w_\ell}{\partial \phi_\ell^2} \left(\frac{\partial \pi_\ell}{\partial \mathbf{x}} \frac{\partial \phi_\ell}{\partial \pi_\ell} \right)^2. \end{aligned}$$

Set $\frac{\partial^2 w_\ell}{\partial \phi_\ell^2} = 0$ so that the continuity of $\nabla^2 w_\ell$ is not spoiled by the discontinuity of $\frac{\partial \pi_\ell}{\partial \mathbf{x}}$. Tracing back, we have $\nabla \hat{d}_i(\mathbf{x}) \in C^1$ and eventually $\nabla d(\mathbf{x}) \in C^1$ as desired.

REFERENCES

Abhishek Madan and David I. W. Levin. 2022. Fast evaluation of smooth distance constraints on co-dimensional geometry. *ACM Transactions on Graphics* 41, 4 (Jul 2022), 1–17. <https://doi.org/10.1145/3528223.3530093>